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Design of Stabilizing Control Laws for Mechanical Systems Based on Lyapunov's Method

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Design methods of stabilizing control laws are discussed for general mechanical systems. Hamiltonians of the mechanical systems are assumed not to depend on time explicitly, but the mechanical systems may embody elastic bodies and nonlinear motions. The methods are based on Lyapunov's second method for stability analysis, and the equations of motion of the controlled systems are not necessary for control design. A candidate for the Lyapunov function is the composition of the Hamiltonians of the subsystems in a given system. A sufficient condition of stabilizing control laws is derived and is extended so as to allow a class of frequency-dependent control laws with positive real filters, observers, or prefilters. The multiplier method (augmented Lagrangian method) is also introduced to attain desirable equilibrium in the closed-loop systems and results in integral control action. Slew maneuver of a flexible beam is employed as a design example to demonstrate the closed-loop characteristics of several control laws based on the present method.

I. Introduction

LYAPUNOV'S second method is a powerful tool for the stability analysis of dynamical systems. One of the preferable features of the method is that it is not necessary to obtain the explicit solutions of the differential equations governing the dynamical systems and, therefore, it can often be applied successfully to nonlinear systems and distributed parameter systems. The method is suitable especially for stability analysis of mechanical systems, since the Hamiltonians of mechanical systems are natural candidates of the Lyapunov function.

Lyapunov's method is employed in Ref. 1 for designing control laws for rigid manipulators that are nonlinear systems. The Lyapunov function consists of the Hamiltonian of the whole system and a potential function introduced to achieve the desirable configuration of the manipulators. Lyapunov's method is utilized in Ref. 2 to analyze the stability of direct velocity feedback for active vibration suppression of flexible space structures, where the system energy is chosen as a candidate of the Lyapunov function. In Ref. 3, flexible multilink manipulators are treated, and the closed-loop stability of proportional-derivative (PD) and strain feedback is analyzed applying Lyapunov's method to an ordinary differential equation obtained through modal expansion. It is shown in Ref. 4 that Lyapunov's method is directly applicable without modal expansion to coupled partial and ordinary differential equations that describe the slew maneuver of a flexible beam, and the PD control law for the attitude angle is shown to be stable by employing the Lyapunov function that consists of the Hamiltonian of the whole system and a potential function to attain a desirable attitude angle. In Refs. 5 and 6, which consider a problem similar to Ref. 4, the Lyapunov function consists of a weighted sum of Hamiltonians of the rigid hub and the flexible beam, not simply the Hamiltonian of the whole system, and a potential function to attain desirable equilibrium. The coupled partial and ordinary differential equations are treated directly without modal expansion in the analysis, and the resultant control law feeds back the attitude angle, the angular rate, the shear force, and the bending moment at the root of the beam. It is shown that the feedback law with a constant gain results in satisfactory attitude control and vibration suppression of the flexible beam. The control laws is modified in Ref. 7 to track a reference trajectory to achieve near-minimum-time maneuvers of a flexible beam. Several attempts are made for generalization[8–11] and simplification[10,12] of the method to treat hybrid systems governed by coupled partial and ordinary differential equations. It is shown in Refs. 10 and 12 that the stabilizing control laws for mechanical systems can be obtained from the work-energy relationship without recourse to equations of motion of controlled systems. The control law in Refs. 5–7 is modified and analyzed in Ref. 13 in terms of momentum exchange rather than the work-energy relationship.

This paper is intended to unify the conventional results based on the work-energy relationship and to make further extensions. The present analysis does not cover results based on momentum exchange[13] or Lyapunov functions that do not consist of the Hamiltonians.[14] The approach in this paper is similar to one in Refs. 10 and 12 in the sense that the control laws are designed without using equations of motion of controlled systems. This paper extends the conventional methods, however, so that the control laws can be frequency dependent, which leads to great variety in control design. It is shown that stabilizing control laws can be designed with positive real filters or observers for generalized velocities of the controlled system. Prefilters are also introduced for the command input of generalized coordinates, and the multiplier method (augmented Lagrangian method) is introduced in the control design regarding the control process as a minimization process that minimizes a candidate of the Lyapunov function subject to an equality constraint. It is shown that the multiplier method results in integral control action.

Several dynamic control laws are studied for mechanical systems in literature especially in the field of robotics. Most control laws require a complete model of the controlled system and are applicable only to control of rigid manipulators.[15] The joint controller considered in Ref. 16 for flexible-link manipulators is restricted to be passive, and controllers in Ref. 17 are restricted to be second order, although they can be adopted in control problems this paper does not consider. In comparison with conventional work on dynamic control laws for mechanical systems, control laws in this paper do not require the complete model of a controlled system and are not restricted to be passive when observers are used. Slew maneuver of a flexible beam is employed as a design example, and simulation results clarify features of the control laws.

II. Stabilizing Control for Mechanical Systems

We derive a sufficient condition for stabilizing a mechanical system based on Lyapunov's method in this section. The condition provides the basis to develop more sophisticated stabilizing control

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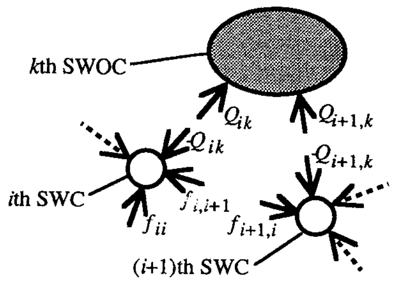

Fig. 1 Free body diagram of a mechanical system.

laws in the next section. We consider a mechanical system whose Hamiltonian does not depend on time explicitly. It is assumed that nonconservative forces in the whole system are dissipative except the control inputs. The system is divided into two classes of subsystems: subsystems with direct control forces (SWC) and subsystems without direct control forces (SWOC). It is assumed that any two SWOC is separated by SWC, i.e., any path from one SWOC to another SWOC passes through at least one SWC. This assumption implies that there is no direct interaction between any two SWOC, which simplifies analysis. If this assumption does not hold, dummy SWC with no control force may be introduced, or some neighboring SWOC have to be unified so as to satisfy the assumption. The division of the system is not unique. The total derivatives of the Hamiltonians with respect to time are given by the products of the generalized forces and the corresponding generalized velocities, as is well known in analytical dynamics. The generalized forces that work on a SWC consist of the control inputs and contributions of constraint forces between the SWC and other subsystems. The total derivative of the Hamiltonian of the ith SWC is expressed as

$$\dot{H}_{ci} = \left(\sum_{j=1}^{n} f_{ij} - \sum_{k=1}^{m} Q_{ik}\right)^T \dot{q} + D_{ci} \qquad (i = 1, 2, \ldots, n) \quad (1)$$

where n and m are the total number of SWC and SWOC, respectively, H_{ci} is the Hamiltonian of the ith SWC, and q is the vector of generalized coordinates that describes all degrees of freedom of all the SWC. Displacements of SWOC are not included in q. The dissipation of energy is nonpositive and is denoted by D_{ci}. The vector f_{ij} is the vector of generalized forces to the ith SWC from the jth SWC and Q_{ik} the vector of generalized forces to the ith SWC from the kth SWOC (see Fig. 1). Elements of f_{ii} represent external forces, and elements of $f_{ij} = -f_{ji} (i \neq j)$ are internal forces. Some elements of f_{ij} can be assigned as the control inputs. Other elements of f_{ij} and all elements of Q_{ik} represent the contributions of constraint forces between two subsystems and cannot be assigned freely. The derivatives of the Hamiltonian H_{fk} of the kth SWOC is expressed as

$$\dot{H}_{fk} = \left(\sum_{i=1}^{n} Q_{ik}\right)^T \dot{q} + D_{fk} \qquad (k = 1, 2, \ldots, m) \quad (2)$$

where D_{fk} is the nonpositive dissipation of energy in the kth SWOC. The Hamiltonian for the whole system H is identical with the sum of the Hamiltonians of all the subsystems, and its total derivative with respect to time is expressed as

$$\dot{H} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}\right)^T \dot{q} + \sum_{i=1}^{n} D_{ci} + \sum_{k=1}^{m} D_{fk} \quad (3)$$

The contributions of constraint forces is canceled in Eq. (3), as is consistent with the fact that constraint forces do no work. Some elements of f_{ij} also vanish in Eq. (3), if they represent contributions of constraint forces between SWC.

To derive stabilizing control laws for the system, a candidate of a Lyapunov function that reflects the control objectives is defined for the system as

$$M = \sum_{i=1}^{n} a_i H_{ci} + \sum_{k=1}^{m} b_k H_{fk} + V(q) \quad (4)$$

where a_i and b_k are scalar constants, and $V(q)$ denotes a generalized potential that is given by a designer. Note that the function M consists of the weighted sum of the Hamiltonians of the subsystems, not simply the Hamiltonian for the whole system. This choice of the candidate of the Lyapunov function results in control laws that include contribution of the constraint forces. The total derivative of M with respect to time is expressed as follows:

$$\dot{M} = \tilde{f}^T \dot{q} + \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} \quad (5)$$

$$\tilde{f} = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_i f_{ij} + \sum_{k=1}^{m}(b_k - a_i) Q_{ik}\right] + \left(\frac{dV}{dq}\right)^T \quad (6)$$

Equation (5) represents a generalization of the work-energy relationship. The Lyapunov stability theorem[18] implies that if free elements in f_{ij} can be chosen so as to satisfy the condition

$$\tilde{f} = -K\dot{q} \quad (7)$$

where K denotes a positive semidefinite matrix, then $dM/dt \leq 0$ and the state that achieves the minimum of M is stable. Therefore stabilizing control laws are designed from Eqs. (6) and (7). The matrix K corresponds to a feedback gain of the generalized velocities, and it is not necessarily symmetric. Note that dM/dt is negative semidefinite and only stability is guaranteed by the Lyapunov theorem. The asymptotic stability of the closed loop is often guaranteed by applying an extension of the Lyapunov theorem.[18] The minimum point of M is asymptotically stable if it is an isolated globally minimum point and if the set of points at which dM/dt is zero contains no positive half-trajectory except the minimum point.

III. Extension of Control Law

Several frequency-dependent control laws are introduced in this section by assigning $\tilde{f}$ appropriately. Lyapunov functions are constructed by adding terms to the original Lyapunov function M. Since the derivative of the Lyapunov function with respect to time is evaluated without the complete model of the controlled system, the advantage of Lyapunov's method is preserved and distributed parameter systems can be treated directly without any discretization.

A. Positive Real Filter

The right-hand side of Eq. (7) can be replaced as

$$\tilde{f} = -y \quad (8)$$

where y denotes the output of a linear time-invariant system with a minimal realization

$$\dot{x}_f = A_f x_f + B_f \dot{q} \quad (9)$$

$$y = C_f x_f + D_f \dot{q} \quad (10)$$

with the transfer function matrix $C_f(sI - A_f)^{-1}B_f + D_f$ positive real. When Eqs. (8–10) are employed in the control design, the sensors outputs of the generalized velocities are filtered first and then fed back to the control inputs. The positive real lemma[19] implies that there exist a symmetric positive definite matrix P_f and matrices L and J such that

$$A_f^T P_f + P_f A_f = -L^T L \quad (11)$$

$$B_f^T P_f + J^T L = C_f \quad (12)$$

$$J^T J = D_f^T + D_f \quad (13)$$

The function M is augmented to M_f as

$$M_f = M + \tfrac{1}{2} x_f^T P_f x_f \quad (14)$$

Under the condition of Eq. (8), the following is shown from direct computation:

$$\dot{M}_f = -\frac{1}{2}\|Lx_f + J\dot{q}\|^2 + \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} \leq 0 \quad (15)$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. It can be concluded from Eq. (15) that the feedback gains of the generalized velocities in stabilizing control laws can be not only constant but also frequency dependent. In a physical sense, the sensors outputs of generalized velocities may be filtered by passive filters without making the closed-loop system unstable. When the transfer function matrix of the filter is strictly proper ($D_f = 0$), the control law can be implemented without the sensors of generalized velocities, since the following holds:

$$C_f(sI - A_f)^{-1}B_f\dot{q}(s) = sC_f(sI - A_f)^{-1}B_f[q(s) - (1/s)q_0]$$

$$= \left[C_f A_f(sI - A_f)^{-1}B_f + C_f B_f\right][q(s) - (1/s)q_0] \quad (16)$$

where q_0 denotes the initial condition of $q(t)$ at $t = 0$. The strictly proper filter for the generalized velocities is replaced by a proper filter for the generalized coordinates, and direct measurement of the generalized velocities is not necessary. Filtering of velocity measurements or using only position measurements is advantageous to control of robots rather than control of spacecraft, since spacecraft are usually equipped with high-precision rate gyros to measure velocities. Although robot systems are usually equipped with high-precision position sensors, velocity measurements by tachometers are often contaminated with a considerable amount of noise. Therefore, direct feedback of velocity measurements is not preferable in control of robots.

B. Observer-Based Control

Next we consider a special case in which the dynamics of only SWC, not the whole system, is expressed as a linear differential equation

$$\ddot{q} = A_o\dot{q} + u \quad (17)$$

where A_o is a constant matrix and u a vector of known quantities. The vector u may include nonlinear terms as far as they are known. If only the generalized coordinate q is measured, the generalized velocities are estimated by a state observer for the following linear system:

$$\dot{x} = Ax + Bu \quad (18)$$

$$q = Cx \quad (19)$$

where

$$x = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}, \qquad A = \begin{bmatrix} A_o & 0 \\ I & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$C = [0 \quad I] \quad (20)$$

The Luenberger observer is designed for the system as

$$\dot{z} = Dz + Eq + Gu \quad (21)$$

$$\hat{x} = L_1 z + L_2 q \quad (22)$$

where $\hat{x}$ denotes the estimate of the state x, the matrix D is stable, and there exists a matrix N such that

$$NA - DN = EC \quad (23)$$

$$L_1 N + L_2 C = I \quad (24)$$

$$G = NB \quad (25)$$

The estimation error converges zero as $t \to \infty$, because it is governed by

$$\hat{x}(t) - x(t) = L_1 e^{Dt}[z(0) - Nx(0)] \quad (26)$$

and D is stable. The estimation error is zero identically, if the initial condition $z(0)$ can be set equal to $Nx(0)$. Stabilizing control laws that utilize the observer are designed by replacing Eq. (7) with

$$\bar{f} = -K\hat{\dot{q}} \quad (27)$$

where $\hat{\dot{q}}$ denotes the vector of the estimates of the generalized velocities given by

$$\hat{\dot{q}} = C_o\hat{x}, \qquad C_o = [I \quad 0] \quad (28)$$

and K is a positive semidefinite constant matrix. The function M is augmented in this case to M_0 defined as

$$M_o = M + \tfrac{1}{2}(z - Nx)^T S(z - Nx) \quad (29)$$

where the matrix S denotes the symmetric positive semidefinite solution for the Lyapunov equation,

$$D^T S + SD = -L_1^T C_o^T R_o C_o L_1 \quad (30)$$

where R_o denotes a symmetric positive semidefinite matrix that can be assigned arbitrarily. The total derivative of M_o with respect to time is expressed as follows:

$$\dot{M}_o = -\frac{1}{2}\begin{bmatrix} \dot{q} \\ e \end{bmatrix}^T \begin{bmatrix} K^T + K & K \\ K^T & R_o \end{bmatrix}\begin{bmatrix} \dot{q} \\ e \end{bmatrix}$$
$$+ \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} \quad (31)$$

where e denotes the estimation error in the generalized velocities

$$e = \hat{\dot{q}} - \dot{q} \quad (32)$$

The stability of the closed-loop system is assured from Eq. (31) if there exists a symmetric positive semidefinite matrix R_o such that

$$\begin{bmatrix} K^T + K & K \\ K^T & R_o \end{bmatrix} \geq 0 \quad (33)$$

When K is symmetric, R_o can be chosen as $R_o = K$. In the general case, the following theorem is useful for testing existence of R_o that satisfies Eq. (33).

Theorem 1. Suppose a symmetric matrix is partitioned as

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \quad (34)$$

where A and C are square and A is symmetric positive semidefinite. There exists a symmetric matrix $C \geq 0$ such that the matrix is positive semidefinite, if and only if the following holds:

$$\mathcal{N}(A) \subset \mathcal{N}(B) \quad (35)$$

where $\mathcal{N}(A)$ denotes the null space of A. Especially, there exists a symmetric matrix $C > 0$ such that the matrix in Eq. (34) is positive definite, if and only if $\mathcal{N}(A) = \{0\}$, i.e., A is symmetric positive definite. The condition of Eq. (35) can also be expressed as follows:

$$\mathcal{N}(A) \subset \mathcal{N}(B) \Leftrightarrow \mathcal{R}(A) \supset \mathcal{R}(B^T) \Leftrightarrow \mathcal{R}([A \quad B^T])$$
$$= \mathcal{R}(A) \Leftrightarrow \operatorname{rank}[A \quad B^T] = \operatorname{rank} A \quad (36)$$

where $\mathcal{R}(A)$ denotes the range space of A. □

Proof of this theorem is presented in the Appendix. The theorem implies that the closed-loop system is stable if the matrix K is positive definite even if it is not symmetric. When K is positive semidefinite, the stability condition is obtained from Eq. (33) and Theorem 1 as

$$\mathcal{N}(K^T + K) \subset \mathcal{N}(K^T) \quad (37)$$

The generalized coordinates are not replaced by their estimates in the control law as shown from Eqs. (6) and (27), and they are assumed to be measured directly. Therefore, the minimal order observer for only the generalized velocities is preferable in the practical point of view. The minimal order observer is designed for the present system as

$$\dot{z} = (A_o - L_o)\hat{q} + u \tag{38}$$

$$\hat{q} = z + L_o q \tag{39}$$

where the matrix L_o is chosen so that $A_o - L_o$ is stable. Note that the dimension of z is identical to the dimension of q. The minimal order observer in Eqs. (38) and (39) corresponds to the observer given by Eqs. (21–25) with

$$D = A_o - L_o, \qquad E = (A_o - L_o)L_o, \qquad G = I \tag{40}$$

$$N = [I \quad -L_o], \qquad L_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} L_o \\ 0 \end{bmatrix} \tag{41}$$

As seen from Eqs. (26) and (41), the estimation error is zero identically, if $\dot{q}(0)$ is known a priori and $z(0)$ is chosen as $z(0) = \dot{q}(0) - L_o q(0)$.

A significant difference between the positive real filter stated previously and the present observer is that the observer needs the dynamical model of the SWC, whereas the positive real filter does not. However, the control laws are designed without the dynamical models of SWOC that may include elastic bodies, and identification of SWC is not as cumbersome as that of SWOC since SWC usually consist of rigid bodies.

C. Prefilter for Command Input

We next consider a control law that feeds back the error between generalized coordinates and reference signals. The reference signals are assumed to be generated by a command input through a stable linear prefilter expressed as follows:

$$\dot{x}_r = A_r x_r + B_r r \tag{42}$$

$$q_r = C_r x_r + D_r r \tag{43}$$

where r is the command input, x_r is the state of the prefilter, and q_r is a vector of the reference signals, respectively. It is assumed that all eigenvalues of A_r have negative real parts. Therefore, A_r is stable and nonsingular. If the command input r is constant, the prefilter has the steady state

$$\bar{x}_r = -A_r^{-1} B_r r \tag{44}$$

$$\bar{q}_r = C_r \bar{x}_r + B_r r \tag{45}$$

To obtain a control law that feeds back the error between generalized coordinates and the reference signals, the generalized potential $V(q)$ is defined as

$$V(q) = \tfrac{1}{2}(q - \bar{q}_r)^T Q (q - \bar{q}_r) \tag{46}$$

and $\tilde{f}$ is given by

$$\tilde{f} = -K\dot{q} - Q(q_r - \bar{q}_r) \tag{47}$$

where K is a positive semidefinite matrix and Q is a symmetric positive semidefinite matrix. Note that $\bar{q}_r$ is eliminated by equating the right-hand sides of Eqs. (6) and (47), and the error $e_r = q - q_r$ remains in the equation for the control inputs. The candidate of the Lyapunov function is defined in this case as

$$M_r = M + \tfrac{1}{2}(x_r - \bar{x}_r)^T P_r (x_r - \bar{x}_r) \tag{48}$$

where P_r denotes a symmetric positive semidefinite matrix that satisfies the following Lyapunov equation:

$$A_r^T P_r + P_r A_r = -C_r^T R_r C_r \tag{49}$$

where R_r denotes a symmetric positive semidefinite matrix that can be assigned arbitrarily. The total derivative of M_r with respect to time is expressed as

$$\dot{M}_r = -\frac{1}{2}\begin{bmatrix} \dot{q} \\ q_r - \bar{q}_r \end{bmatrix}^T \begin{bmatrix} K^T + K & Q \\ Q & R_r \end{bmatrix} \begin{bmatrix} \dot{q} \\ q_r - \bar{q}_r \end{bmatrix} + \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} \tag{50}$$

Theorem 1 is applicable to the quadratic form in Eq. (50). There exists $R_r \geq 0$ such that M_r is monotonously decreasing if the following holds:

$$\mathcal{N}(K^T + K) \subset \mathcal{N}(Q) \tag{51}$$

D. Combination of Observer, Positive Real Filter, and Prefilter

The preceding results can be extended to such a case that the estimates of the generalized velocities are filtered by a positive real filter and the prefilter is used. The generalized velocities $\dot{q}$ in Eqs. (9) and (10) are replaced by their estimates $\hat{\dot{q}}$ generated by Eqs. (21) and (22). The candidate of the Lyapunov function is constructed by augmenting Eqs. (14), (29), and (48) as follows:

$$M_{rof} = M + \tfrac{1}{2}x_f^T P_f x_f + \tfrac{1}{2}(z - Nx)^T S(z - Nx) + \tfrac{1}{2}(x_r - \bar{x}_r)^T P_r (x_r - \bar{x}_r) \tag{52}$$

The total derivative of M_{rof} with respect to time is obtained as

$$\dot{M}_{rof} = -\frac{1}{2}\begin{bmatrix} x_f \\ \dot{q} \\ e \\ e_r \end{bmatrix}^T \begin{bmatrix} L^T L & L^T J & -P_f B_f & 0 \\ J^T L & J^T J & D_f & Q \\ -B_f^T P_f & D_f^T & R_o & 0 \\ 0 & Q & 0 & R_r \end{bmatrix} \times \begin{bmatrix} x_f \\ \dot{q} \\ e \\ e_r \end{bmatrix} + \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} \tag{53}$$

Theorem 1 is applicable to the quadratic form in Eq. (53). There exists block-diag$(R_o, R_r) \geq 0$ such that M_{rof} is monotonously decreasing if the condition

$$\mathcal{N}\left(\begin{bmatrix} L^T L & L^T J \\ J^T L & J^T J \end{bmatrix}\right) \subset \mathcal{N}\left(\begin{bmatrix} -B_f^T P_f & D_f^T \\ 0 & Q \end{bmatrix}\right) \tag{54}$$

is satisfied. The condition (54) is further simplified because the following holds:

$$\mathcal{N}\left(\begin{bmatrix} L^T L & L^T J \\ J^T L & J^T J \end{bmatrix}\right) = \mathcal{N}([L \quad J]) \tag{55}$$

E. Multiplier Method

The control process discussed in this paper can be regarded as a minimization process of the function M. One drawback of the present approach is that the minimum point of the function M is not assignable freely in general. With the presence of conservative external forces, the minimum point of M may not be identical with that of $V(q)$ assigned by a designer. If the equilibrium of the closed-loop system can be predicted from the dynamical model, the function $V(q)$ may be able to be chosen appropriately so that the function M has a desirable minimum point. In the case of rigid manipulators discussed in Ref. 1, the potential energy in the Hamiltonian depends on only the generalized coordinate q of SWC, and the minimum point of M can be assigned freely with respect to q by defining an artificial potential function $V(q)$ appropriately. In mechanical systems with elastic bodies, however, the potential energy depends not only on q but also on elastic displacements of SWOC. Therefore the function $V(q)$ cannot be chosen so as to cancel the potential energy completely, and the minimum point of M depends on the elastic displacement. It is a complicated task to choose the function $V(q)$ so that the function M has a desired minimum point with respect to q. Some algorithms for adjusting $V(q)$ are desirable in order to design the feedback laws with simplicity.

A control objective to achieve the equality

$$h(q) = 0 \tag{56}$$

is equivalent to a minimization problem of M subject to the equality constraint, Eq. (56). A minimization problem with the equality constraint is converted to an unconstrained minimization problem through use of the multiplier method[20] (augmented Lagrangian method). The constrained minimization problem is solved by successively solving a series of unconstrained minimization problems that minimize the function M with $V(q)$ defined as

$$V_i(q) = \lambda_i^T h + (c_1/2)h^T h \qquad (i = 1, 2, 3, \ldots) \tag{57}$$

where λ_i is the estimate of the multiplier and c_1 is a sufficiently large positive constant. Note that there is a penalty term in Eq. (57) that distinguishes the multiplier method from the simple Lagrange's method of multipliers. The multiplier is updated by the following formula:

$$\lambda_{i+1} = \lambda_i + c_1 h \tag{58}$$

and λ_1 is given arbitrarily ($\lambda_1 = 0$ is often chosen). The resultant control law is such that the control is executed for a fixed $V_i(q)$, and $V_i(q)$ is updated successively when the system responses approach the equilibrium sufficiently close.

Continuous control laws can also be designed in contrast to the control law based on Eqs. (57) and (58). The generalized potential V is defined as a time-dependent function,

$$V(q, t) = \lambda^T(t)h(q) + (c_1/2)h^T(q)h(q) \tag{59}$$

where c_1 is a positive constant that is sufficiently large and $\lambda(t)$ is the multiplier. Motivated by that the update formula Eq. (58) can be regarded as a kind of difference equation, the update formula is naturally modified in the present method to the differential equation,

$$\dot{\lambda} = c_2 h(q); \qquad \lambda(0) = 0 \tag{60}$$

where c_2 is a positive constant. The difference between c_1 and c_2 reflects the scaling of time. The partial derivative of V with respect to the generalized coordinates is obtained from Eqs. (59) and (60) as

$$\frac{\partial V}{\partial q} = c_2\left(\int_0^t h\,\mathrm{d}\tau\right)^T \frac{\mathrm{d}h}{\mathrm{d}q} + c_1 h^T \frac{\mathrm{d}h}{\mathrm{d}q} \tag{61}$$

The multiplier method results in integral control action as shown in Eq. (61). Integral control action reduces the steady-state error as is well known in servo theory. Expression of the identical constraint in a different form leads to a different control law and, consequently, different performance. The total derivative of the function M is expressed in this case as

$$\dot{M} = \tilde{f}^T \dot{q} + \sum_{i=1}^{n} a_i D_{ci} + \sum_{k=1}^{m} b_k D_{fk} + \frac{\partial V}{\partial t} \tag{62}$$

and the closed-loop stability is not assured from $\mathrm{d}M/\mathrm{d}t$ since

$$\frac{\partial V}{\partial t} = c_2 h^T h \geq 0 \tag{63}$$

holds. The integral gain c_2 should be small enough so that the closed-loop system is not unstable.

IV. Example

A. System Model and Control Design

1. Conventional Control Law

Slew maneuver of a flexible beam is a suitable model to demonstrate the present approach with proper simplicity and is employed as a numerical example in this paper. The system is modeled as a planar motion of a rigid hub equipped with a flexible beam as illustrated in Fig. 2. The control objectives are to control the attitude angle of the rigid hub and to suppress the vibration of the flexible beam. The control torque τ is applied at the center of rotation O. Several control laws are designed based on the present approach. Dissipation in the system may be neglected in the control design, since it does not affect the stability characteristics significantly. The SWC is the rigid hub, and the SWOC is the flexible beam in the system. The boundary between the two subsystems can be different from the root of the beam as far as the constraint forces at the boundary can be measured, though we select the present boundary in order to design control laws that are similar to the conventional one in Refs. 5 and 6. The degree of freedom of the rigid hub is described by the attitude angle θ. The total derivative of the Hamiltonian of the rigid hub is obtained evidently from Fig. 3 as

$$\dot{H}_{c1} = [\tau - (l_0 S_0 - M_0)]\dot{\theta} \tag{64}$$

where l_0 is the radius of the hub and S_0 and M_0 are shear force and bending moment at the root of the beam, respectively. The term $l_0 S_0 - M_0$ represents the contribution of the constraint forces between the hub and the beam. The total derivative of the Hamiltonian of the flexible beam is expressed as

$$\dot{H}_{f1} = (l_0 S_0 - M_0)\dot{\theta} \tag{65}$$

The derivative of the Hamiltonian for the whole system is expressed as

$$\dot{H} = \dot{H}_{c1} + \dot{H}_{f1} = \tau\dot{\theta} \tag{66}$$

The candidate of the Lyapunov function M is defined in this case as follows:

$$M = aH_{c1} + bH_{f1} + (c/2)\theta^2 \tag{67}$$

where a, b, and c are positive constants. The third term in the right-hand side of Eq. (67) is introduced in order to achieve the desirable attitude angle. The desirable attitude angle is set to zero without loss of generality. The total derivative of M with respect to time is obtained from Eqs. (64), (65), and (67) as

$$\dot{M} = \tilde{\tau}\dot{\theta} \tag{68}$$

$$\tilde{\tau} = a\tau + (b - a)(l_0 S_0 - M_0) + c\theta \tag{69}$$

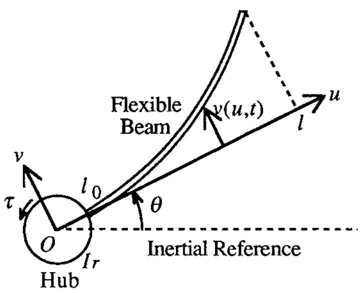

Fig. 2 Slew maneuver model.

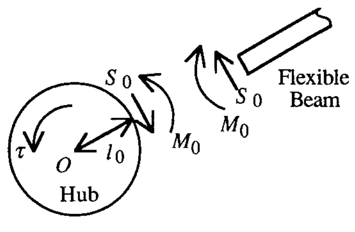

Fig. 3 Free body diagram of slew maneuver.

A stabilizing control law is obtained by choosing the control torque so as to satisfy

$$\tilde{\tau} = -k\dot{\theta} \tag{70}$$

where k is a positive constant. The resultant control law is of the same form as in Refs. 5 and 6,

$$\tau = -(1/a)[c\theta + k\dot{\theta} + (b-a)(l_0 S_0 - M_0)] \tag{71}$$

The contribution of the constraint forces $l_0 S_0 - M_0$ vanishes in the control law if $b = a$, i.e., the Hamiltonians of the hub and the flexible beam are weighted equally. The inclusion of S_0 and M_0 in the control law has an important effect on the vibration suppression of the flexible beam.[5,6]

2. Positive Real Filter

The analysis in Sec. III.A shows that the k in Eq. (71) may be a positive-real transfer function. A low-pass filter is chosen as $k(s)$ in the numerical example as

$$k(s) = k_1/(1+sT) \tag{72}$$

where k_1 is a positive constant and T is the time constant of the filter. The transfer function Eq. (72) is a strictly positive-real function.

3. Observer-Based Control

If the angular rate $\dot{\theta}$ is not measured directly, it has to be estimated by an observer. Only the dynamical model of the rigid hub is necessary in order to design the observer. The dynamics of the hub is governed by a differential equation,

$$\ddot{\theta} = (1/I_r)[\tau - (l_0 S_0 - M_0)] \tag{73}$$

where I_r is the moment of inertia of the rigid hub, and the right-hand side consists of known quantities in the control. The minimal-order observer and the control law are given by

$$\dot{z} = -L_o\hat{\dot{\theta}} + (1/I_r)[\tau - (l_0 S_0 - M_0)] \tag{74}$$

$$\hat{\dot{\theta}} = z + L_o\theta \tag{75}$$

$$\tau = -(1/a)[c\theta + k\hat{\dot{\theta}} + (b-a)(l_0 S_0 - M_0)] \tag{76}$$

where L_o is a positive constant and $\hat{\dot{\theta}}$ is the estimate of $\dot{\theta}$, The preceding equations are rearranged to obtain the stabilizing control law in the form of a first-order dynamic compensator,

$$\dot{z} = -\left(L_o + \frac{k}{aI_r}\right)z - \left(\frac{c+kL_o}{aI_r} + L_o^2\right)\theta - \frac{b}{aI_r}(l_0 S_0 - M_0) \tag{77}$$

$$\tau = -(1/a)[(c+kL_o)\theta + kz + (b-a)(l_0 S_0 - M_0)] \tag{78}$$

The initial condition of $z(t)$ should be set as $z(0) = \dot{\theta}(0) - L_o\theta(0)$ if $\dot{\theta}(0)$ is known a priori, which is often a case in the slew maneuver.

4. Prefilter for Command Input

Analysis in Sec. III.D results in a control law of the following form:

$$\tau = -(1/a)[c(\theta - \theta_r) + k\dot{\theta} + (b-a)(l_0 S_0 - M_0)] \tag{79}$$

where the reference angle θ_r is generated by a stable second-order prefilter in this example:

$$\theta_r(s) = \frac{1}{\left(s^2 + 2\zeta\omega_o + \omega_o^2\right)} r(s) \qquad (\zeta > 0) \tag{80}$$

The condition of Eq. (51) is satisfied in this case, and the closed-loop stability is assured if the command input $r(t)$ is a step function. Note that the control law Eq. (79) does not include reference signals corresponding to the angular velocity, the shear force, and the bending moment, whereas the tracking control law in Ref. 7 does.

5. Multiplier Method (Integral Control Action)

It is evident when there is no external disturbance that the equilibrium of the closed-loop system is given by $\theta = 0, \dot{\theta} = 0, S_0 = 0$, and $M_0 = 0$ without displacement in the flexible beam. When a constant external force of unknown magnitude is applied at a point on the flexible beam, the external force can be expressed in terms of a potential function defined as

$$U = [v(u_f) + u_f\theta]f \tag{81}$$

where f is the constant external force of unknown magnitude, u_f the point where the external force is applied, and $v(u_f)$ the displacement of the beam at that point. The external force can be regarded as a conservative force if the potential function U is included in the Hamiltonian of the flexible beam. The potential U does not have minimum with respect to the coordinates $v(u_f)$ and θ, and the minimum point of the function M differs from that in the case without the external force. Therefore, integral control action discussed in Sec. III.E is necessary to attain the same equilibrium in the closed loop as in the case without the external force. The equality constraint to be achieved is given in this example as

$$\theta = 0 \tag{82}$$

and the modified control law is designed as

$$\tau = -\frac{1}{a}\left[c_1\theta + c_2\int_0^t \theta\,\mathrm{d}\tau + k\dot{\theta} + (b-a)(l_0 S_0 - M_0)\right] \tag{83}$$

where c_1 and c_2 are positive constants. The closed-loop stability is not guaranteed for the modified control law due to integral control action. The constant c_1 should be sufficiently large as is required in the multiplier method, and c_2 should be small enough so that the closed-loop system is not unstable.

B. Simulation Results

Numerical simulation of the slew maneuver is carried out in order to demonstrate the control laws designed in the preceding subsection. The parameters of the model are set identical to those of the experiment in Ref. 21. Damping terms are included in the numerical simulation. A state space model is derived in terms of vehicle modes truncated by the forth flexible mode. Several parameters in the control laws are fixed through the simulation as $a = 1, b = 21, c = c_1 = 0.3, k = k_1 = 0.55$. The flexible beam is at the equilibrium state, and attitude angle is equal to -60 deg ($-\pi/3 = -1.05$ rad) without angular rate at the initial time $t = 0$.

Closed-loop responses of the control law Eq. (71) with the constant gain are shown in Fig. 4. The figure shows the attitude angle $\theta(t)$, the angular rate of the attitude angle $\dot{\theta}(t)$, the bending moment at the root of the beam $M_0(t)$, and the control torque $\tau(t)$, respectively. The attitude angle is successfully controlled and vibration in the beam is suppressed by the simple control law.

The low-pass filter of the angular rate, Eq. (72), is employed in the responses of Fig. 5. The time constant of the filter is chosen as $T = 0.5$ that results in the cut-off frequency at 0.32 Hz to demonstrate the effect of the filter. The cut-off frequency of the filter is higher than the natural frequency of the first flexible mode and is lower than the natural frequency of the second flexible mode. It is shown in Fig. 5 that the higher flexible modes are not controlled successfully, though the low-frequency behavior seems similar to the responses in Fig. 4. It may be concluded that the feedback of the angular rate is essential not only to control of attitude angle but also to vibration suppression of the flexible beam. The responses in Fig. 4 is restored with sufficiently high cut-off frequency of the filter.

Closed-loop responses of the control law Eqs. (77) and (78) are shown in Fig. 6. The control law feeds back the estimate of the angular rate. The parameter L_o is chosen as $L_o = 1$ in the control law. Though the initial angular rate is known to be zero in this example, the initial condition $z(0)$ is set to zero so that the initial estimate error is not equal to zero and the responses are different from those in Fig. 4. The same responses are obtained as in Fig. 4 if the initial condition $z(0)$ is chosen as $z(0) = -L_o\theta(0)$. The estimate

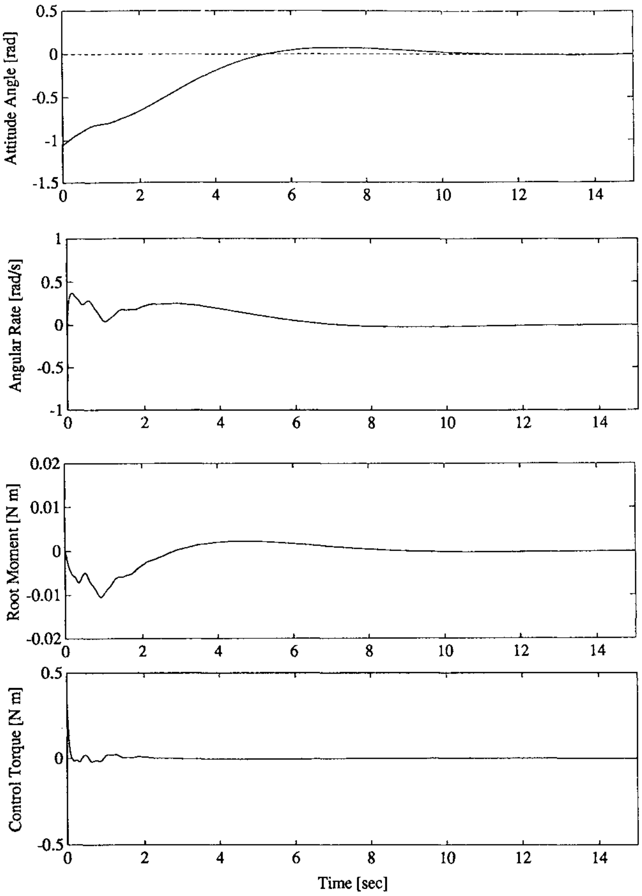

Fig. 4 Closed-loop responses of the constant-gain feedback.

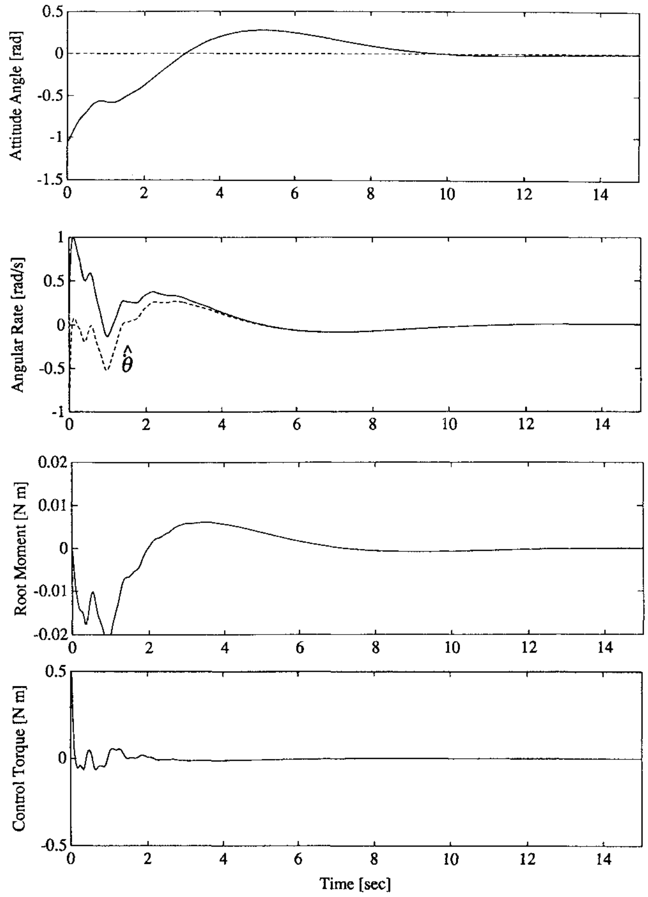

Fig. 6 Closed-loop responses of the control law with a minimal-order observer.

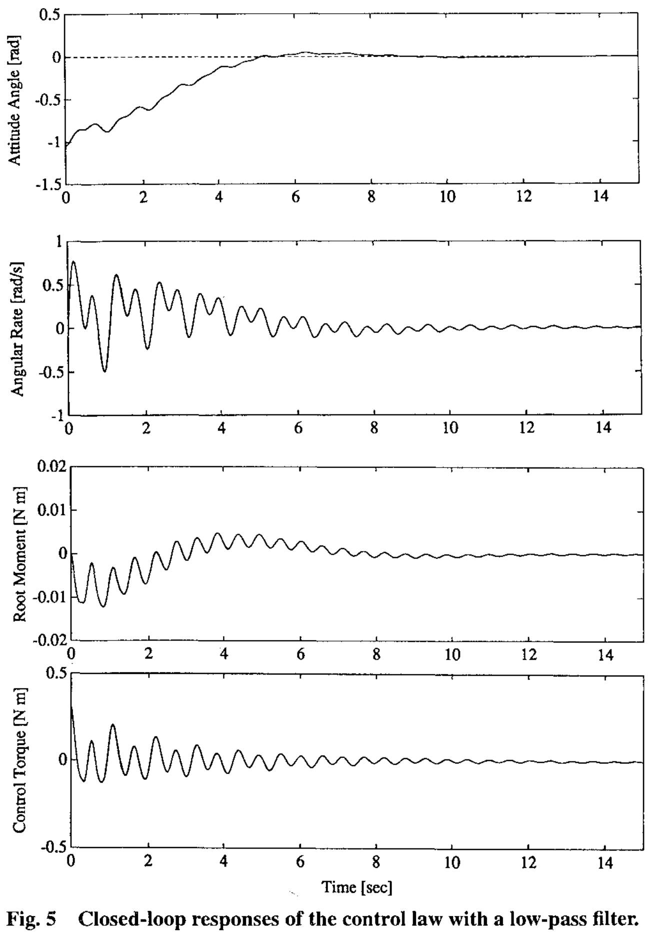

Fig. 5 Closed-loop responses of the control law with a low-pass filter.

of the angular rate is plotted as the broken line superposed on the actual angular rate in Fig. 6. The estimate converges to the actual value as the time increases. The overshoot in the attitude angle and the maximum amplitude of the angular rate, the root moment, and the control torque are larger than those in Fig. 4, since the estimation error leads to the excessive control torque as is seen at the initial time.

Closed-loop responses of the control law in Eq. (79) is shown in Fig. 7. The reference angle θ_r is given by Eq. (80) with $\zeta = 0.7$ and $\omega_o = 1$ rad/s, and the command input r is given as a step function of the magnitude $r(t) = \pi/3\,(t \geq 0)$. The attitude angle converges to the reference angle slowly since the control law does not include reference signals corresponding to the angular velocity, the shear force, and the bending moment. The main advantage of the present control law is its small amplitude of the control torque rather than the tracking of the reference signal. Since the reference angle increases smoothly, the error angle is small at the initial time. Consequently, the control law does not generate the impulsive torque at the initial time, and the vibration of high frequency is not excited in the flexible beam.

When an constant external force, $f = -0.01$ N, is applied at the middle point of the flexible beam, i.e., $u_f = 0.5(l + l_0)$, the control law Eq. (71) with the constant gain does not bring the attitude angle to zero, as shown in Fig. 8. The closed-loop system has a different equilibrium from the desired state. If the control law is modified to Eq. (83), which includes integral control action, the desired attitude angle is achieved in spite of the external force, as shown in Fig. 9. The feedback gain c_2 for integral control action is selected as $c_2 = 0.04$ in the simulation. The appropriate value of c_2 is found through trial and error. A relevant difference between Figs. 8 and 9 is seen only in the attitude angle. The attitude angle attains the desired value in Fig. 9 because of integral control action. In both cases, the steady-state control torque is $-fu_f$ to balance the torque caused by the external force. The flexible beam has displacement at the equilibrium of the closed-loop system, and the root moment converges to a nonzero value in Figs. 8 and 9.

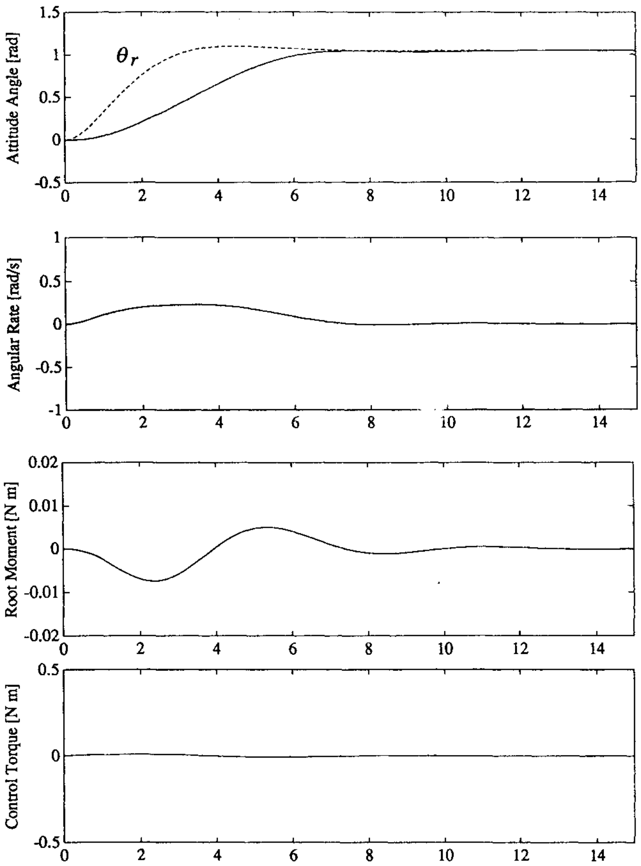

Fig. 7 Closed-loop responses of the control law with a prefilter for command input.

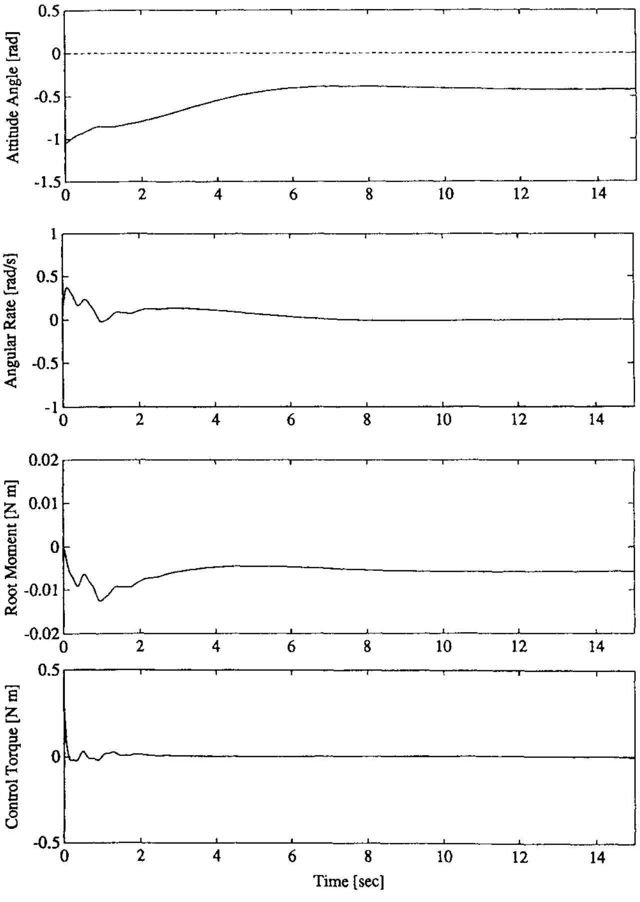

Fig. 8 Closed-loop responses of the constant-gain feedback; a constant external force is applied on the flexible beam.

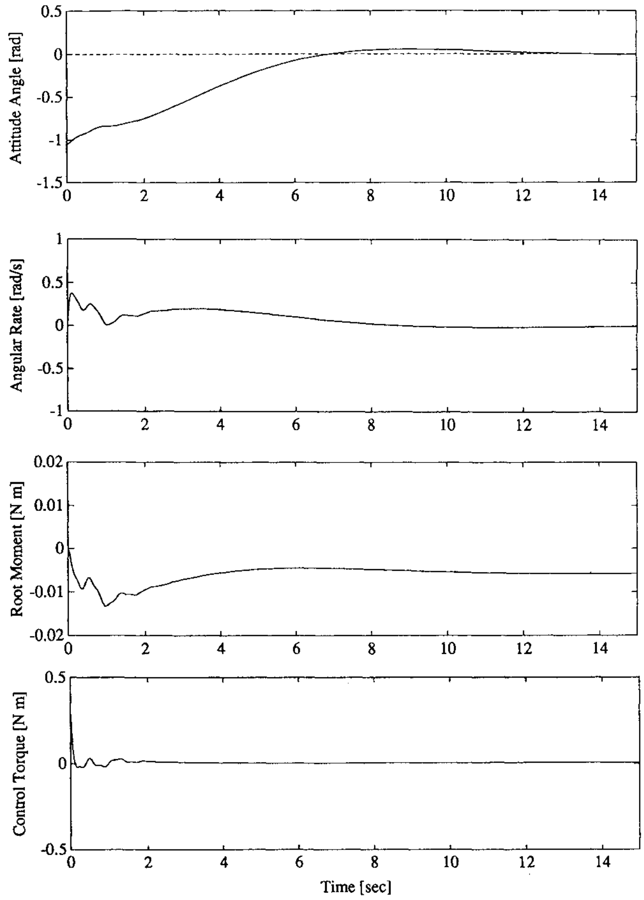

Fig. 9 Closed-loop responses of the control law with integral control action; a constant external force is applied on the flexible beam.

V. Conclusions

Design methods of stabilizing control laws are discussed for general mechanical systems in this paper. The methods are based on Lyapunov's second method for stability analysis, and the equations of motion are not necessary for control design. A candidate of the Lyapunov function is the composition of the Hamiltonians of the subsystems in a given system. Hamiltonians of the mechanical systems are assumed not to depend on time explicitly, but the mechanical systems may embody elastic bodies and nonlinear motions. A sufficient condition of the stabilizing control laws is derived from the Lyapunov stability theorem, and further analysis shows that the closed loop is stable with such a class of frequency-dependent control laws that have positive-real filters, observers for the generalized velocities, prefilters for the command input, or a combination thereof. The multiplier method is also introduced to attain desirable equilibrium in the closed-loop systems and results in integral control action. Slew maneuver of a flexible beam is employed as a design example for the present methods, and the simulation results show closed-loop characteristics of several control laws. Further research is recommended on design of frequency-dependent control laws with better performance, since they have more variety than the control laws of constant gain.

Appendix: Proof of Theorem 1

Since Eq. (36) is apparent, the necessity and sufficiency of Eq. (35) is proved. We denote the quadratic form by

$$g(\boldsymbol{x}_1, \boldsymbol{x}_2) = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}^T \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \tag{A1}$$

(If) If $\mathcal{N}(A) \subset \mathcal{N}(B)$, then the vector $\boldsymbol{x}_1$ is expressed as

$$\boldsymbol{x}_1 = \boldsymbol{x} + \boldsymbol{y}, \qquad \boldsymbol{x} \in \mathcal{N}(A) \subset \mathcal{N}(B), \qquad \boldsymbol{y} \in \mathcal{N}(A)^{\perp} \tag{A2}$$

where $\mathcal{N}(A)^{\perp}$ is the orthogonal complement of $\mathcal{N}(A)$, and x and y are determined uniquely for a given x_1. A modification of the Rayleigh–Ritz theorem[22] implies

$$\min_{\substack{y \neq 0 \\ y \in \mathcal{N}(A)^{\perp}}} (y^T A y / y^T y) = \lambda_p \tag{A3}$$

where $\lambda_p (>0)$ denotes the minimum of nonzero eigenvalues of A. Using Eq. (A3) and the fact that $Ax = Bx = 0$, we have

$$\begin{aligned} g(x_1, x_2) &= y^T A y + 2x_2^T B y + x_2^T C x_2 \\ &\geq \lambda_p \|y\|^2 - 2\sigma_{\max}(B) \|y\| \|x_2\| + x_2^T C x_2 \\ &= \left(\sqrt{\lambda_p} \|y\| - \frac{\sigma_{\max}(B)}{\sqrt{\lambda_p}} \|x_2\| \right)^2 \\ &\quad + x_2^T \left(C - \frac{\sigma_{\max}(B)^2}{\lambda_p} I \right) x_2 \end{aligned} \tag{A4}$$

where $\sigma_{\max}(B)$ denotes the maximum singular value of B. Therefore, $g(x_1, x_2)$ is positive semidefinite by choosing C such that $C \geq [\sigma_{\max}(B)^2/\lambda_p] I \geq 0$. Especially, if $\mathcal{N}(A) = \{0\}$, then $x_1 = y$, and $g(x_1, x_2)$ is positive definite by choosing C such that $C > [\sigma_{\max}(B)^2/\lambda_p] I$.

(Only if) If $\mathcal{N}(A) \subset \mathcal{N}(B)$ does not hold, then there exists a vector z such that $Az = 0$ and $Bz \neq 0$. For any symmetric positive semidefinite matrix C, we have

$$g(z, kBz) = 2k\|Bz\|^2 + k^2 z B^T C B z \leq k[2 + k\lambda_{\max}(C)] \|Bz\|^2 \tag{A5}$$

where k is a scalar real number and $\lambda_{\max}(C) (\geq 0)$ is the maximum eigenvalue of C. By choosing k with sufficiently small magnitude such that $k < 0$ and $2 + k\lambda_{\max}(C) > 0$, $g(z, kBz)$ is negative, i.e., $g(x_1, x_2)$ is not positive semidefinite. Therefore, if $g(x_1, x_2)$ is positive semidefinite, $\mathcal{N}(A) \subset \mathcal{N}(B)$ has to hold. Especially, if $g(x_1, x_2)$ is positive definite, then $g(x_1, 0) = x_1^T A x_1$ is positive for any nonzero x_1, which implies $\mathcal{N}(A) = \{0\}$. □

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